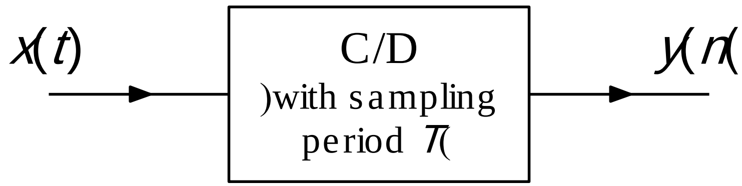
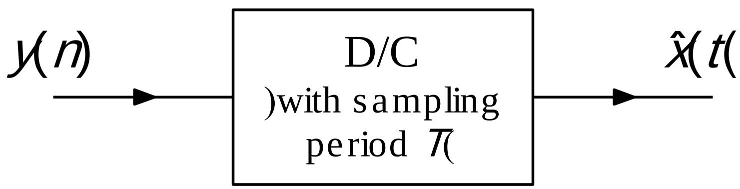
Section 5.10

Application: Sampling and Interpolation

- Often, we want to be able to *convert* between continuous-time and discrete-time representations of a signal.
- This is accomplished through processes known as *sampling* and *interpolation*.
- The *sampling* process, which is performed by an ideal continuous-time to discrete-time (C/D) converter shown below, transforms a continuous-time signal x to a discrete-time signal (i.e., sequence) y.



- The *interpolation* process, which is performed by an ideal discrete-time to continuous-time (D/C) converter shown below, transforms a discrete-time signal y to a continuous-time signal $\hat{x}$.

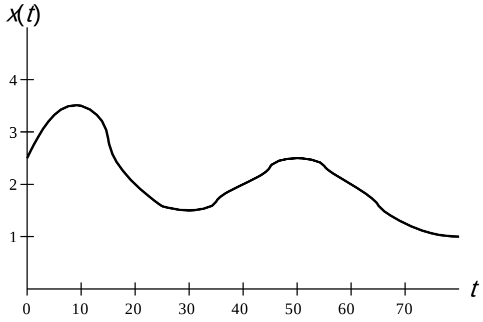


- Note that, unless very special conditions are met, the sampling process loses information (i.e., is *not invertible*).

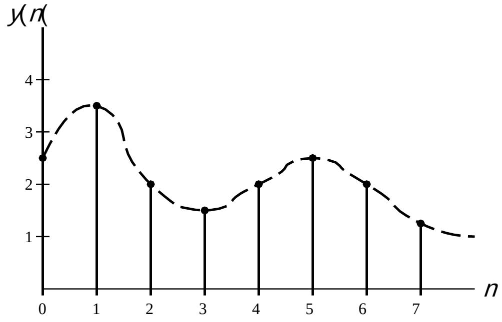
- Although sampling can be performed in many different ways, the most commonly used scheme is periodic sampling.
- With this scheme, a sequence y of samples is obtained from a continuous-time signal x according to the relation

$$y(n) = x(nT) \quad \text{for all integer } n,$$

where T is a positive real constant.
- As a matter of terminology, we refer to T as the sampling period, and $\omega_s = 2\pi/T$ as the (angular) sampling frequency.
- An example of periodic sampling is shown below, where the original continuous-time signal x has been sampled with *sampling period* $T = 10$, yielding the sequence y.



Original Signal



Sampled Signal

- The sampling process is not generally invertible.
- In the absence of any constraints, a continuous-time signal cannot usually be uniquely determined from a sequence of its equally-spaced samples.
- Consider, for example, the continuous-time signals x_1 and x_2 given by

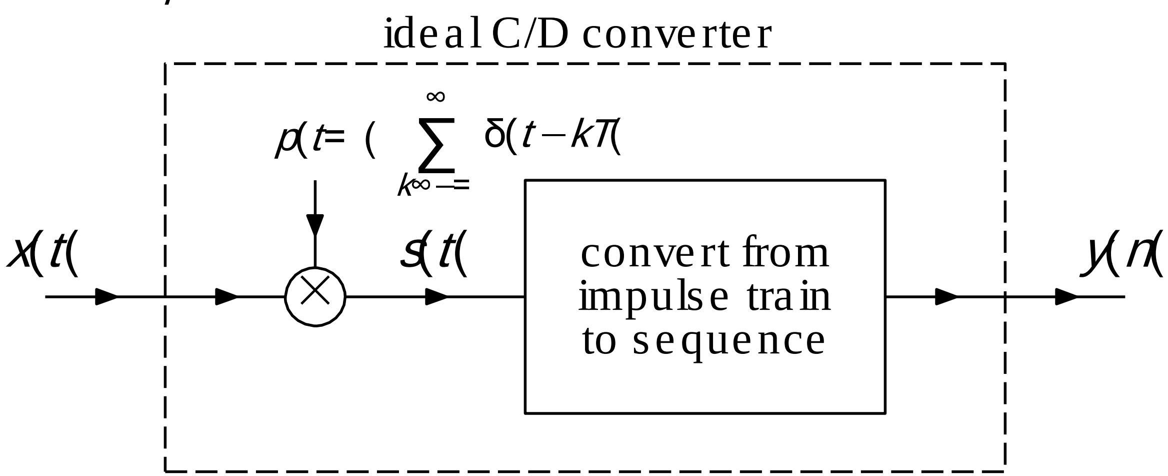
$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

- If we sample each of these signals with the sampling period $T = 1$, we obtain the respective sequences

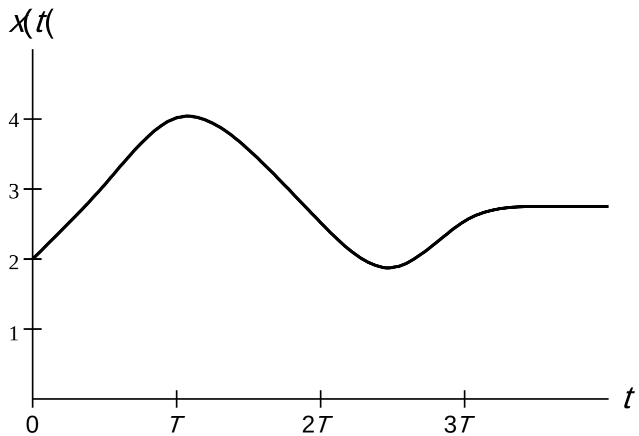
$$y_1(n) = x_1(nT) = x_1(n) = 0 \quad \text{and}$$
$$y_2(n) = x_2(nT) = \sin(2\pi n) = 0.$$

- Thus, $y_1(n) = y_2(n)$ for all n, although $x_1(t) \neq x_2(t)$ for all noninteger t.
- Fortunately, under certain circumstances, a continuous-time signal can be recovered exactly from its samples.

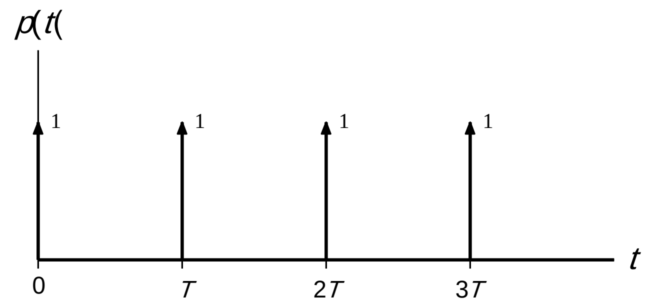
- An impulse train is a signal of the form $v(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t - kT)$, where a_k and T are real constants (i.e., $v(t)$ consists of weighted impulses spaced apart by T).
- For the purposes of analysis, sampling with sampling period T and frequency $\omega_s = \frac{2\pi}{T}$ can be modelled as shown below.



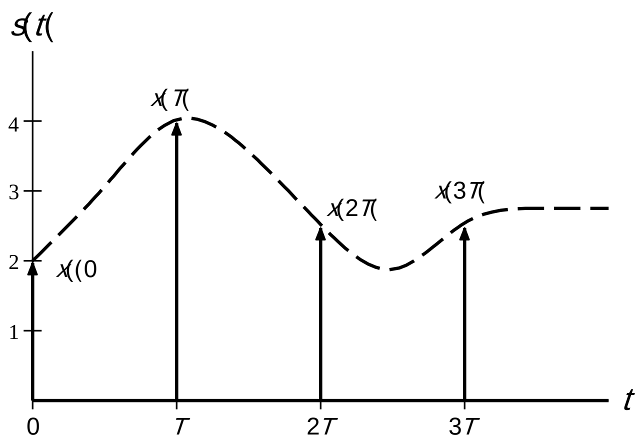
- The sampling of a continuous-time signal x to produce a sequence y consists of the following two steps (in order):
  1. Multiply the signal x to be sampled by a periodic impulse train p, yielding the impulse train s.
  2. Convert the impulse train s to a sequence y, by forming a sequence from the weights of successive impulses in s.



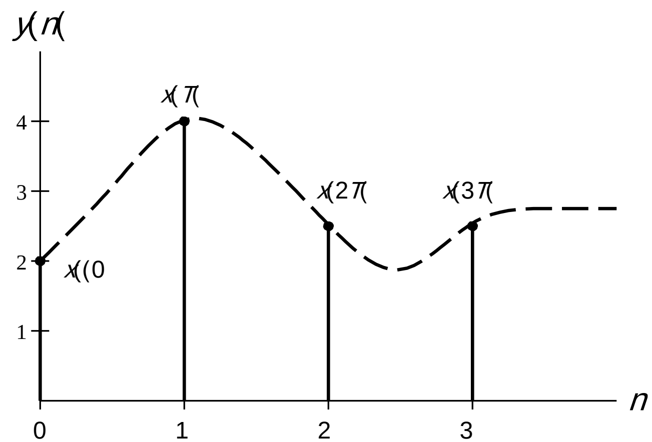
Input Signal (Continuous-Time(



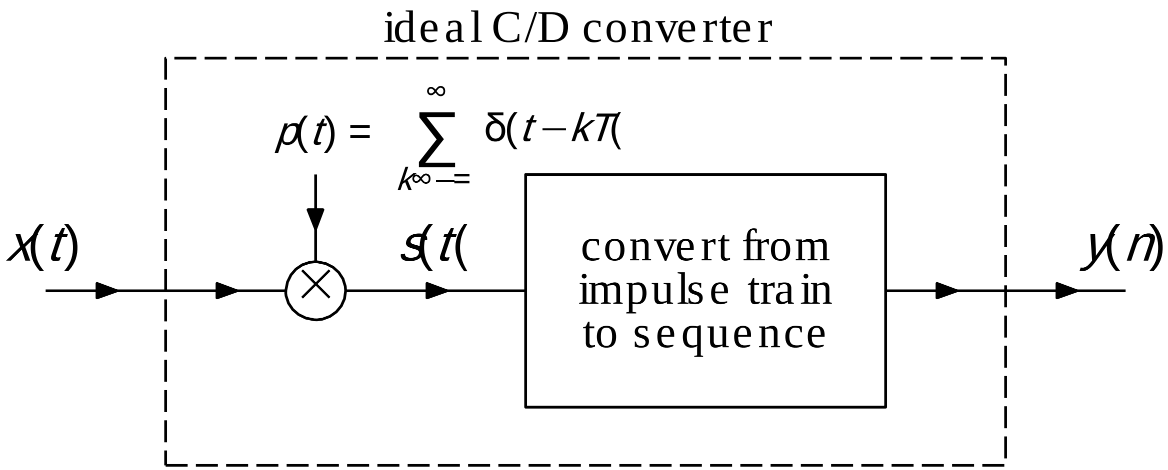
Periodic Impulse Train



Impulse-Sampled Signal
)Continuous-Time(



Output Sequence *(Discrete-Time(*



- In the time domain, the impulse-sampled signal s is given by

$$s(t) = x(t)\,p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT).$$

- In the Fourier domain, the preceding equation becomes

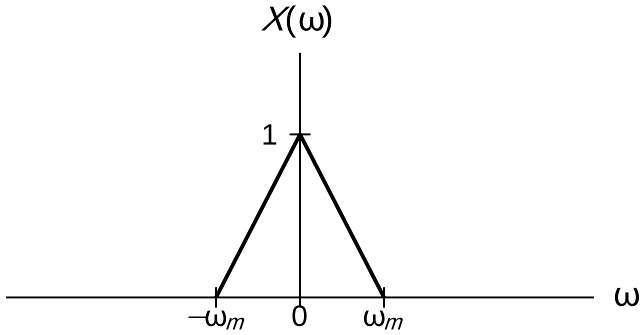
$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

- Thus, the spectrum of the impulse-sampled signal s is a scaled sum of an infinite number of *shifted copies* of the spectrum of the original signal x.

- Consider frequency spectrum S of the impulse-sampled signal s given by

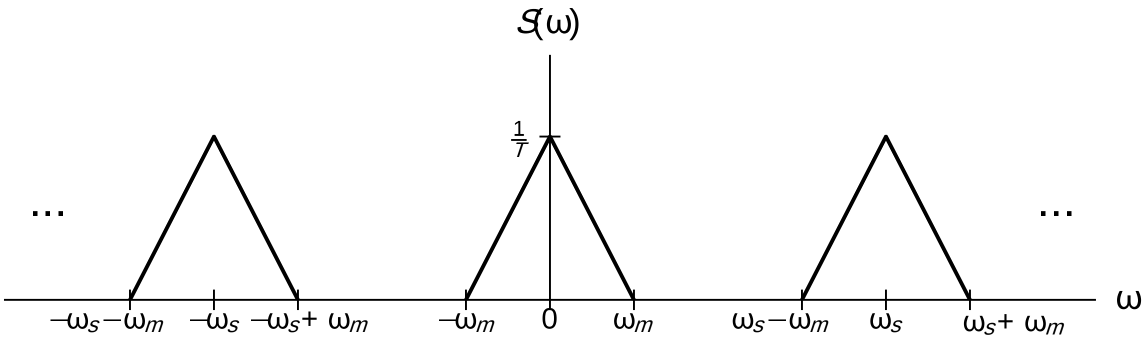
$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

- The function S is a scaled sum of an infinite number of *shifted copies* of X.
- Two distinct behaviors can result in this summation, depending on ω_s and the bandwidth of x.
- In particular, the nonzero portions of the different shifted copies of X can either:
  1. overlap; or
  2. not overlap.
- In the case where overlap occurs, the various shifted copies of X add together in such a way that the original shape of X is lost. This phenomenon is known as aliasing.
- When aliasing occurs, the original signal x cannot be recovered from its samples in y.



Spectrum of Input Signal

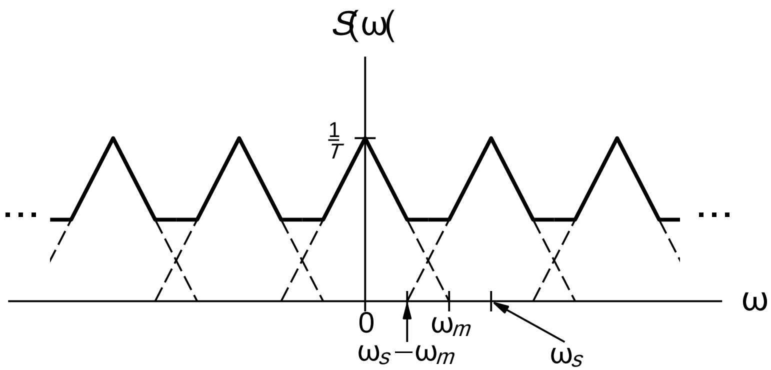
(Bandwidth ω_m)



Spectrum of Impulse-Sampled Signal:

No Aliasing Case

)$\omega_s > 2\omega_m$(

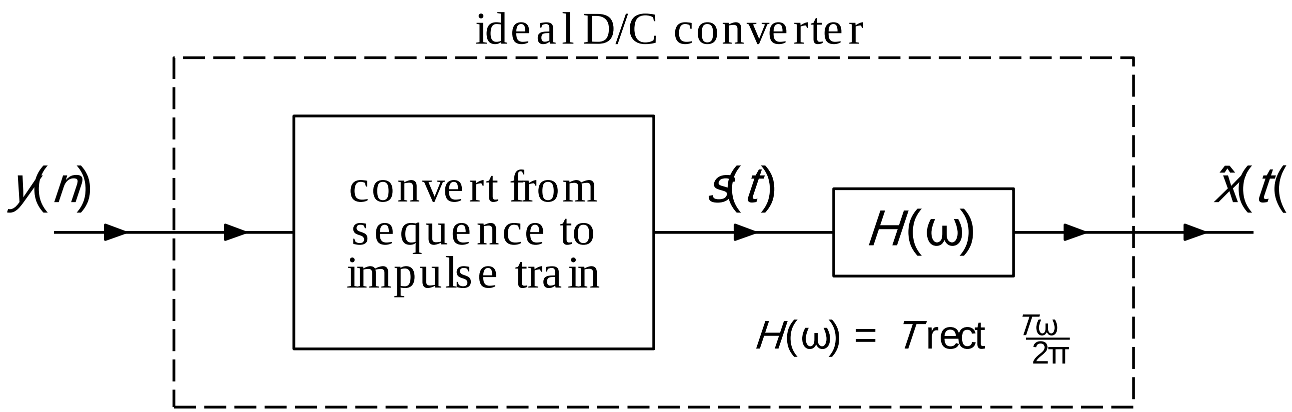


Spectrum of Impulse-Sampled Signal:

Aliasing Case

($\omega_s \leq 2\omega_m$(

- For the purposes of analysis, interpolation can be modelled as shown below.



- The inverse Fourier transform h of H is $h(t) = \text{sinc}(\pi t/T)$.
- The reconstruction of a continuous-time signal x from its sequence y of samples (i.e., bandlimited interpolation) consists of the following two steps (in order):
  1. Convert the sequence y to the impulse train s, by using the elements in the sequence as the weights of successive impulses in the impulse train. Apply a
  2. lowpass filter to s to produce $\hat{x}$.
- The lowpass filter is used to eliminate the extra copies of the original signal's spectrum present in the spectrum of the impulse-sampled signal s.

- In more detail, the reconstruction process proceeds as follows.
- First, we convert the sequence y to the impulse train s to obtain

$$s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t-nT).$$

- Then, we filter the resulting signal s with the lowpass filter having impulse response h, yielding

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} y(n)\operatorname{sinc}\left(\tfrac{\pi}{T}(t-nT)\right).$$

- **Sampling Theorem.** Let x be a signal with Fourier transform X, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., x is bandlimited to frequencies $[-\omega_M, \omega_M]$). Then, x is uniquely determined by its samples $y(n) = x(nT)$ for all integer n, if

$$\omega_s > 2\omega_M,$$

where $\omega_s = 2\pi/T$. The preceding inequality is known as the Nyquist condition. If this condition is satisfied, we have that

$$x(t) = \sum_{n=-\infty}^{\infty} y(n)\,\text{sinc}\left(\tfrac{\pi}{T}(t - nT)\right),$$

or equivalently (i.e., rewritten in terms of ω_s instead of T),

$$x(t) = \sum_{n=-\infty}^{\infty} y(n)\,\text{sinc}\left(\tfrac{\omega_s}{2}t - \pi n\right).$$

- We call $\omega_s/2$ the Nyquist frequency and $2\omega_M$ the Nyquist rate.

Part 6

Laplace Transform (LT(

- Another important mathematical tool in the study of signals and systems is known as the Laplace transform.
- The Laplace transform can be viewed as a *generalization of the Fourier transform*.
- Due to its more general nature, the Laplace transform has a number of *advantages* over the Fourier transform.
- First, the Laplace transform representation exists for some signals that do not have Fourier transform representations. So, we can handle a *larger class of signals* with the Laplace transform.
- Second, since the Laplace transform is a more general tool, it can provide *additional insights* beyond those facilitated by the Fourier transform.

- Earlier, we saw that complex exponentials are eigenfunctions of LTI systems.
- In particular, for a LTI system $\mathcal{H}$ with impulse response h, we have that

$$\mathcal{H}\{e^{st}\} = H(s)e^{st} \quad \text{where} \quad H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt.$$

- Previously, we referred to H as the system function.
- As it turns out, H is the Laplace transform of h.
- Since the Laplace transform has already appeared earlier in the context of LTI systems, it is clearly a useful tool.
- Furthermore, as we will see, the Laplace transform has many additional uses.